

A LOCAL LIMIT THEOREM FOR CLOSED GEODESICS AND HOMOLOGY

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ABSTRACT. In this paper, we study the distribution of closed geodesics on a compact negatively curved manifold. We concentrate on geodesics lying in a prescribed homology class and, under certain conditions, obtain a local limit theorem to describe the asymptotic behaviour of the associated counting function as the homology class varies.

0. INTRODUCTION

Let M be a compact smooth Riemannian manifold with first Betti number $k > 0$ and with negative sectional curvatures. Suppose also that either $\dim M = 2$ or that M is $1/4$ -pinched, i.e., the sectional curvatures all lie in an interval $[-\kappa, -\kappa/4]$, for some $\kappa > 0$. Such a manifold contains a countable infinity of prime closed geodesics. (We say that a closed geodesic is prime if it is not a nontrivial multiple of another closed geodesic.) In this paper we are interested in how these closed geodesics are distributed with respect to homology.

The homology group $H_1(M, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^k \oplus \text{Tor}$, where Tor is the (finite) torsion subgroup. In this paper, it will be convenient to consider the torsion-free part of the homology, $H_1(M, \mathbb{Z})/\text{Tor}$. We shall, in fact, assume that an isomorphism has been fixed and write \mathbb{Z}^k instead of $H_1(M, \mathbb{Z})/\text{Tor}$.

For a typical (prime) closed geodesic γ on M , let $l(\gamma)$ denote its length and $[\gamma] \in H_1(M, \mathbb{Z})/\text{Tor} = \mathbb{Z}^k$ the torsion-free part of its homology class. For $\alpha \in \mathbb{Z}^k$, define a counting function

$$\pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\}.$$

Recently, several papers have studied the asymptotics of this function as $T \rightarrow \infty$. In particular, Anantharaman [1] and Pollicott and Sharp [17] have shown that there exist constants $C_0 > 0$, independent of α , and $C_n(\alpha)$, $n \geq 1$, such that, for any $N \geq 1$, we have the asymptotic expansion

$$(0.1) \quad \pi(T, \alpha) = \frac{e^{hT}}{T^{k/2+1}} \left(C_0 + \frac{C_1(\alpha)}{T} + \frac{C_2(\alpha)}{T^2} + \cdots + \frac{C_N(\alpha)}{T^N} + O\left(\frac{1}{T^{N+1}}\right) \right),$$

where $h > 0$ denotes the topological entropy of the geodesic flow over M . (In fact, the expansion in [17] contains some extra terms corresponding to powers of $T^{-1/2}$; a more careful analysis, as carried out in [11], shows that these terms vanish.) Furthermore, Kotani [11], has studied the dependence of the coefficients $C_n(\alpha)$ on

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$\alpha = (\alpha_1, \dots, \alpha_k)$, showing that they may be expressed as polynomials of degree $2n$ in $\alpha_1, \dots, \alpha_k$. In the special case of manifolds of constant negative curvature, the expansion (0.1) was obtained by Phillips and Sarnak [14] and, independently, Katsuda and Sunada [9] obtained the leading term. For manifolds of variable negative curvature (without the pinching condition) the leading term of (0.1) was obtained by Katsuda [8], Lalley [12] and Pollicott [15]. Analogous results for manifolds with cusps have been obtained by Epstein [6] and Babillot and Peigné [3].

In this note, we take a slightly different view and address the question of the behaviour of $\pi(T, \alpha)$ when α is allowed to vary independently of T . We obtain the following “local limit theorem”.

Theorem 1. *Let M be a compact smooth Riemannian manifold with first Betti number $k > 0$ and with negative sectional curvatures. Suppose also that either $\dim M = 2$ or that M is $1/4$ -pinched. Then there exists a symmetric positive definite real matrix \mathcal{D} such that*

$$\lim_{T \rightarrow \infty} \left| \frac{h\sigma^k T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1} \alpha \rangle / 2T} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^k$, where $\sigma > 0$ satisfies $\sigma^{2k} = \det \mathcal{D}$.

Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product $\langle x, y \rangle = x_1 y_1 + \dots + x_k y_k$. As a particular consequence, we recover the leading term of the expansion (0.1), with $C_0 = h^{-1} \sigma^{-k} (2\pi)^{-k/2}$. Theorem 1 appears not to have been stated even for manifolds of constant negative curvature; although, in that case, the result can be easily deduced from the analysis contained in [14].

Remarks. (i) If we take the torsion part of $H_1(M, \mathbb{Z})$ into account, then we need to modify Theorem 1 to read

$$\lim_{T \rightarrow \infty} \left| \frac{h\sigma^k T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{1}{(\#\text{Tor}) (2\pi)^{k/2}} e^{-\langle \alpha_F, \mathcal{D}^{-1} \alpha_F \rangle / 2T} \right| = 0,$$

uniformly in $\alpha \in H_1(M, \mathbb{Z})$, where $\alpha_F \in \mathbb{Z}^k$ denotes the torsion-free part of $\alpha \in H_1(M, \mathbb{Z})$.

(ii) In Kotani’s formula for the term $C_n(\alpha)/T^n$ in (0.1), the highest power of α makes a contribution

$$\frac{1}{(2\pi)^{k/2} h \sigma^k} \frac{1}{n!} \left(-\frac{\langle \alpha, \mathcal{D}^{-1} \alpha \rangle}{2T} \right)^n.$$

As observed by Kotani in [11], formally summing these contributions gives the expression $e^{-\langle \alpha, \mathcal{D}^{-1} \alpha \rangle / 2T} / (2\pi)^{k/2} h \sigma^k$.

Theorem 1 should be compared with the results on homology classes varying linearly in T obtained by Lalley [12] and Babillot and Ledrappier [2]. Using these results, one can show that, for $\delta > 0$ sufficiently small,

$$(0.2) \quad \lim_{T \rightarrow \infty} \sup_{\|\alpha\| \leq \delta T} \left| \frac{T^{k/2+1}}{C(\alpha/T) e^{H(\alpha/T)T}} \pi(T, \alpha) - 1 \right| = 0,$$

where $H(x)$ is an entropy function satisfying $H(0) = h$, $\nabla H(0) = 0$ and $\nabla^2 H(0) = -\mathcal{D}^{-1}$, and where $C(x)$ is continuous with $C(0) = C_0$. On the other hand, Theorem

1 is equivalent to

$$(0.3) \quad \lim_{T \rightarrow \infty} \sup_{\|\alpha\| \leq \delta T} \left| \frac{h\sigma^k T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1} \alpha \rangle / 2T} \right| = 0$$

(as the supremum over $\|\alpha\| > \delta T$ clearly tends to zero). However, even though $H(x) = H(0) - \langle x, \mathcal{D}^{-1} x \rangle / 2 + O(\|x\|^3)$, which gives

$$\exp\{H(\alpha/T)T\} = \exp\{H(0)T - \langle \alpha, \mathcal{D}^{-1} \alpha \rangle / 2T + O(\|\alpha\|^3/T^2)\},$$

the presence of the third order terms means that one cannot deduce (0.3) from (0.2).

The results of [2] and [12] do imply a central limit theorem: for $A \subset \mathbb{R}^k$,

$$\lim_{T \rightarrow \infty} \frac{\#\{\gamma : l(\gamma) \leq T, [\gamma]/\sqrt{T} \in A\}}{\#\{\gamma : l(\gamma) \leq T\}} = \frac{1}{(2\pi)^{k/2} \sigma^k} \int_A e^{-\langle x, \mathcal{D}^{-1} x \rangle / 2} dx.$$

A key ingredient in the proof of Theorem 1 is an understanding of the analytic domain of a family of functions of a complex variable, called L -functions, indexed by the characters of \mathbb{Z}^k . In the next section, we shall define these functions and discuss their properties. In Section 2, we shall introduce a family of functions $\mathcal{S}_T(t)$, $t \in [-\pi, \pi]^k$, obtained by summing a suitably weighted character $e^{i\langle t, \cdot \rangle}$ over all (multiple) closed geodesics of length at most T , and show that they are related to contour integrals of the corresponding L -functions. The results in Section 1 are then used to estimate the sums $\mathcal{S}_T(t)$. In Section 3, we shall use an approach adapted from [18] to transfer information from the $\mathcal{S}_T(t)$ to an auxiliary function $\psi(T, \alpha)$, which is essentially a weighted version of $\pi(T, \alpha)$, and obtain an analogue of Theorem 1 valid for $\psi(T, \alpha)$. In Section 4, we shall complete the proof of Theorem 1 by elementary arguments. In the final section, we shall discuss the application of our method to homologically full Anosov flows, giving a new proof of the first order asymptotic formula for $\pi(T, \alpha)$ (but without uniformity) in that case.

Notation. For given functions $A(T)$ and $B(T)$, we shall write $A(T) \sim B(T)$, as $T \rightarrow \infty$, if $\lim_{T \rightarrow \infty} A(T)/B(T) = 1$, and $A(T) = O(B(T))$ if $|A(T)| \leq CB(T)$, for some constant $C > 0$.

1. L -FUNCTIONS

In order to obtain our main result, we shall need to understand the analytic behaviour of a certain family of functions of a complex variable. We will identify the character group of \mathbb{Z}^k with $[-\pi, \pi]^k$. For $t \in [-\pi, \pi]^k$, define

$$L(s, t) = \prod_{\gamma} \left(1 - e^{-sl(\gamma) + i\langle t, [\gamma] \rangle} \right)^{-1},$$

where the product is taken over all prime closed geodesics γ . This converges for $\operatorname{Re}(s) > h$ and has a meromorphic extension to a strictly larger half-plane [13].

It will be convenient to consider multiple closed geodesics $\gamma' = \gamma^n$, $n \geq 1$. In this case we shall write $l(\gamma') = nl(\gamma)$, $[\gamma'] = n[\gamma]$, and $\Lambda(\gamma') = l(\gamma)$. (Note that Λ is analogous to the von Mangoldt function in number theory.)

We are interested in the logarithmic derivative $L'(s, t)/L(s, t)$ of $L(s, t)$. Whenever the summation converges, we have the identity

$$\frac{L'(s, t)}{L(s, t)} = - \sum_{\gamma'} \Lambda(\gamma') e^{-sl(\gamma') + i\langle t, [\gamma'] \rangle}.$$

We shall make use of the properties of $L'(s, t)/L(s, t)$ described by the following two propositions. These results were obtained in [17] and rely heavily on the techniques of Dolgopyat [4]. We write $U(\delta) = \{t : \|t\| < \delta\}$.

Proposition 1 ([17]). *For all sufficiently small $\delta > 0$ the following statements are true.*

- (i) *There exists $\epsilon > 0$ and an analytic function $s : U(\delta) \rightarrow (-\infty, h]$, satisfying $s(0) = h$ and $s(t) < h$ for $t \neq 0$, such that*

$$\frac{L'(s, t)}{L(s, t)} + \frac{1}{s - s(t)}$$

is analytic in $\operatorname{Re}(s) > h - \epsilon$.

- (ii) *There exists $\epsilon > 0$ such that, for $t \notin U(\delta)$, $L'(s, t)/L(s, t)$ is analytic in $\operatorname{Re}(s) > h - \epsilon$.*

Proposition 2 ([17]). *There exists $\epsilon > 0$, $C > 0$, and $0 < \beta < 1$, such that, for all $t \in [-\pi, \pi]^k$,*

$$\left| \frac{L'(s, t)}{L(s, t)} \right| \leq C |\operatorname{Im}(s)|^\beta,$$

for $\operatorname{Re}(s) > 1 - \epsilon$ and $|\operatorname{Im}(s)| \geq 1$.

The function $s(t)$ enjoys the following properties.

Lemma 1. $\nabla s(0) = 0$ and $\nabla^2 s(0)$ is strictly negative definite.

We shall write $\mathcal{D} = -\nabla^2 s(0)$ and define $\sigma > 0$ by $\sigma^{2k} = \det \mathcal{D}$. The next result is crucial for our subsequent analysis.

Proposition 3. *There exists $\delta > 0$ such that, for $t \in U(\delta\sigma\sqrt{T})$,*

$$\lim_{T \rightarrow \infty} e^{(s(t/\sigma\sqrt{T}) - h)T} = e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2}.$$

Furthermore, $|e^{(s(t/\sigma\sqrt{T}) - h)T}| \leq e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2}$ and

$$\left| e^{(s(t/\sigma\sqrt{T}) - h)T} - e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2} \right| \leq 2e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2}.$$

Proof. Let $f(t) = e^{s(t) - h}$. Then $f(0) = 1$, $\nabla f(0) = \nabla s(0) = 0$, and $\nabla^2 f(0) = \nabla^2 s(0) = -\mathcal{D}$. Applying Taylor's Theorem, we have that, for $\|t/\sigma\sqrt{T}\| \leq \delta$,

$$f\left(\frac{t}{\sigma\sqrt{T}}\right) = 1 - \frac{\langle t, \mathcal{D}t \rangle}{2\sigma^2 T} + O\left(\frac{\|t\|^3}{T^{3/2}}\right)$$

(where the implied constant is independent of t). The first statement follows from the identity $\lim_{T \rightarrow \infty} (1 - x/T)^T = e^{-x}$.

Provided $\delta > 0$ is sufficiently small, for $\|u\| \leq \delta$, we have

$$\langle u, \mathcal{D}u \rangle / 2 + O(\|u\|^3) \geq \langle u, \mathcal{D}u \rangle / 4.$$

Since $(1 - x/T)^T < e^{-x}$, this gives us $|f(t/\sigma\sqrt{T})| \leq e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2}$. Applying the triangle inequality, we obtain

$$\begin{aligned} |f(t/\sigma\sqrt{T}) - e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2}| &\leq e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2} + e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2} \\ &\leq 2e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2}. \end{aligned}$$

Remark. The function $s(t)$ has an interpretation in terms of the thermodynamic formalism of the geodesic flow on SM . For a continuous function $G : SM \rightarrow \mathbb{R}$, define its pressure $P(G) = \sup_{\mu} \{h_{\mu}(\phi) + \int G d\mu\}$, where the supremum is taken over all probability measures invariant under the geodesic flow. We can define a (smooth) function $F : SM \rightarrow \mathbb{R}^k$ with the property that, for each closed geodesic γ , $\int_0^{l(\gamma)} F(\gamma(t), \dot{\gamma}(t)) dt = [\gamma]$. Then $\mathbb{R}^k \ni z \mapsto P(\langle z, F \rangle)$ is real analytic and has an analytic extension to a neighbourhood of \mathbb{R}^k in \mathbb{C}^k . We have that $s(t) = P(\langle it, F \rangle)$ and that $\mathcal{D} = \nabla^2 P(\langle z, F \rangle)|_{z=0}$ [10], [19].

2. CONTOUR INTEGRATION

We shall now use the results on L -functions obtained in the preceding section to examine the behaviour of the summatory function

$$\mathcal{S}_T(t) = \sum'_{l(\gamma') \leq T} \Lambda(\gamma') e^{i\langle t, [\gamma'] \rangle},$$

as $T \rightarrow \infty$. (Here, the $'$ on the summation sign denotes that the terms with $l(\gamma') = T$ are counted with weight $1/2$.)

We begin by relating $\mathcal{S}_T(t)$ to $L'(s, t)/L(s, t)$. This is achieved through the following lemma.

Lemma 2 ([20, p. 132], Effective Perron Formula). *Define a function $\theta(y)$ by*

$$\theta(y) = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{2} & \text{if } y = 1, \\ 1 & \text{if } y > 1. \end{cases}$$

Then, uniformly for $d > 0$, $R > 0$,

$$\left| \theta(y) - \frac{1}{2\pi i} \int_{d-iR}^{d+iR} \frac{y^s}{s} ds \right| = O\left(\frac{y^d}{1 + R|\log y|}\right).$$

Set $d = h + T^{-1}$ and $R = T^K$ (where $K > 0$ will be chosen later). Applying Lemma 2 term-by-term to $-L'(s, t)/L(s, t)$, we obtain

$$(2.1) \quad \mathcal{S}_T(t) = \frac{1}{2\pi i} \int_{d-iR}^{d+iR} \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds + O\left(\sum_{\gamma'} \frac{\Lambda(\gamma') e^{dT} e^{-dl(\gamma')}}{1 + R|T - l(\gamma')|} \right).$$

We will estimate the big- O term in this expression. First set $\epsilon = T^{-M}$ (where $M > 0$ will be chosen later) and consider the terms for which $|T - l(\gamma')| \leq \epsilon$. We will use the following result contained in [16].

Proposition 4 ([16]). *There exists $c < h$ such that*

$$\#\{\gamma' : l(\gamma') \leq T\} = \int_2^{e^{hT}} \frac{1}{\log u} du + O(e^{cT}).$$

As a consequence, we may write

$$\#\{\gamma' : |T - l(\gamma')| \leq \epsilon\} = \int_{e^{hT-h\epsilon}}^{e^{hT+h\epsilon}} \frac{1}{\log u} du + O(e^{cT}) = O\left(\frac{\epsilon e^{hT}}{T}\right).$$

Furthermore, if $|T - l(\gamma')| \leq \epsilon$, then $e^{dT} e^{-dl(\gamma')} = O(1)$. Thus,

$$\sum_{|T-l(\gamma')| \leq \epsilon} \frac{\Lambda(\gamma') e^{dT} e^{-dl(\gamma')}}{1 + R|T - l(\gamma')|} = O\left(\frac{e^{hT}}{T^M}\right).$$

On the other hand,

$$\sum_{|T-l(\gamma')| > \epsilon} \frac{\Lambda(\gamma') e^{dT} e^{-dl(\gamma')}}{1 + R|T - l(\gamma')|} \leq \frac{e^{dT}}{R\epsilon} \sum_{\gamma'} \Lambda(\gamma') e^{-dl(\gamma')} = O\left(\frac{e^{hT}}{T^{K-M-1}}\right),$$

where we have used the estimate

$$\left| \frac{L'(h + T^{-1}, 0)}{L(h + T^{-1}, 0)} \right| = O(T).$$

Combining the estimates above, equation (2.1) becomes

$$(2.2) \quad \mathcal{S}_T(t) = \frac{1}{2\pi i} \int_{d-iR}^{d+iR} \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds + O\left(\frac{e^{hT}}{T^{\min\{M, K-M-1\}}}\right).$$

Lemma 3. *For all $N \geq 1$ we have the following estimates. (The implied constants are independent of t .)*

(i) *For $t \in U(\delta)$,*

$$\mathcal{S}_T(t) = \frac{e^{s(t)T}}{s(t)} + O\left(\frac{e^{hT}}{T^N}\right).$$

(ii) *For $t \notin U(\delta)$,*

$$\mathcal{S}_T(t) = O\left(\frac{e^{hT}}{T^N}\right).$$

Proof. Choose $h - \epsilon < c < h$ and let Γ denote the contour formed by the rectangle with vertices at $d - iR$, $d + iR$, $c + iR$, and $c - iR$, oriented counterclockwise.

(i) Suppose that $t \in U(\delta)$. By Proposition 1(i) we can choose $c < s(t)$ so that, using the Residue Theorem,

$$\frac{1}{2\pi i} \int_{\Gamma} \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds = \frac{e^{s(t)T}}{s(t)}.$$

Using Proposition 2, we also have the following bounds:

(a)

$$\left| \left(\int_{c+iR}^{d+iR} + \int_{c-iR}^{d-iR} \right) \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds \right| = O(R^{\beta-1} e^{hT}) = O\left(\frac{e^{hT}}{T^{K(1-\beta)}}\right);$$

(b)

$$\left| \int_{c-iR}^{c+iR} \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds \right| = O(R^{\beta} e^{cT}) = O(T^{\beta K} e^{cT}).$$

Combining this with (2.2) gives

$$\mathcal{S}_T(t) = \frac{e^{s(t)T}}{s(t)} + O\left(\frac{e^{hT}}{T^N}\right),$$

where

$$N = \min\{M, K - M - 1, K(1 - \beta)\}.$$

Since K and M are arbitrary, we may take N as large as we please.

(ii) Suppose that $t \notin U(\delta)$. Then, by Proposition 1(ii),

$$\frac{1}{2\pi i} \int_{\Gamma} \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{sT}}{s} ds = 0.$$

The result now follows as in the proof of (i).

3. AN AUXILIARY FUNCTION

In this section, we shall prove a result analogous to Theorem 1 but where $\pi(T, \alpha)$ is replaced by the auxiliary function

$$\psi(T, \alpha) = \sum'_{\substack{l(\gamma') \leq T \\ [\gamma'] = \alpha}} \Lambda(\gamma'),$$

which can be related to the sums $\mathcal{S}_T(t)$ considered in the previous section. We shall adapt an approach used by Rousseau-Egele [18] to examine the quantity $\sigma^k T^{k/2} e^{-hT} \psi(T, \alpha)$. For $a > 0$, write $I(a) = [-a, a]^k$. Using the orthogonality relationship

$$\frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i\langle t, \alpha \rangle} e^{i\langle t, y \rangle} dt = \begin{cases} 1 & \text{if } y = \alpha, \\ 0 & \text{if } y \in \mathbb{Z}^k \setminus \alpha, \end{cases}$$

we have that

$$\psi(T, \alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi)} e^{-i\langle t, \alpha \rangle} \mathcal{S}_T(t) dt.$$

Making the substitution $t \mapsto t/\sigma\sqrt{T}$, we obtain

$$\sigma^k T^{k/2} \psi(T, \alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi\sigma\sqrt{T})} e^{-i\langle t, \alpha \rangle / \sigma\sqrt{T}} \mathcal{S}_T(t/\sigma\sqrt{T}) dt.$$

The next result is the key to the proof of Theorem 1.

Proposition 5.

$$\lim_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \frac{h\sigma^k T^{k/2}}{e^{hT}} \psi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2T} \right| = 0.$$

Using the identity

$$e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2T} = \frac{1}{(2\pi)^{k/2}} \int_{\mathbb{R}^k} e^{i\langle t, \alpha \rangle / \sigma\sqrt{T}} e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2} dt,$$

we have established the bound

$$\begin{aligned} & (2\pi)^k \left| \frac{h\sigma^k T^{k/2}}{e^{hT}} \psi(T, \alpha) - \frac{e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2T}}{(2\pi)^{k/2}} \right| \\ & \leq \left| \int_{U(\delta\sigma\sqrt{T})} e^{-i\langle t, \alpha \rangle / \sigma\sqrt{T}} \left\{ h e^{-hT} \mathcal{S}_T(t/\sigma\sqrt{T}) - e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2} \right\} dt \right| \\ & + \left| \int_{I(\pi\sigma\sqrt{T}) \setminus U(\delta\sigma\sqrt{T})} e^{-i\langle t, \alpha \rangle / \sigma\sqrt{T}} h e^{-hT} \mathcal{S}_T(t/\sigma\sqrt{T}) dt \right| \\ & + \left| \int_{\mathbb{R}^k \setminus U(\delta\sigma\sqrt{T})} e^{-i\langle t, \alpha \rangle / \sigma\sqrt{T}} e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2} dt \right| \\ & = A_1(T, \alpha) + A_2(T, \alpha) + A_3(T, \alpha). \end{aligned}$$

An easy calculation shows that

$$\lim_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} A_3(T, \alpha) = 0,$$

so, to complete the proof of Proposition 5, it remains to estimate $A_1(T, \alpha)$ and $A_2(T, \alpha)$. To do this we shall use the information on $s(t)$ and $\mathcal{S}_T(t)$ contained in Proposition 3 and Lemma 3.

Lemma 4. *There exists $C > 0$ such that, for all sufficiently small $\delta > 0$,*

$$\lim_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} A_1(T, \alpha) \leq C \left\{ \int_{\mathbb{R}^k} e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2} dt \right\} \delta^2.$$

Proof. By Lemma 3, we have that, for $t \in U(\delta\sigma\sqrt{T})$,

$$he^{-hT} \mathcal{S}_T(t/\sigma\sqrt{T}) = \frac{he^{(s(t/\sigma\sqrt{T})-h)T}}{s(t/\sigma\sqrt{T})} + O(T^{-(k/2+1)}).$$

Using the analyticity of $s(t)$ and the fact that $\nabla s(0) = 0$, we have

$$\left| e^{(s(t/\sigma\sqrt{T})-h)T} \left(\frac{h}{s(t/\sigma\sqrt{T})} - 1 \right) \right| \leq C\delta^2 e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2},$$

for some constant $C > 0$. Thus,

$$\begin{aligned} A_1(T, \alpha) &\leq \int_{U(\delta\sigma\sqrt{T})} \left| e^{(s(t/\sigma\sqrt{T})-h)T} - e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2} \right| dt \\ &\quad + C\delta^2 \int_{U(\delta\sigma\sqrt{T})} e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2} dt + O\left(\frac{1}{T}\right). \end{aligned}$$

By Proposition 3, we know that $e^{(s(t/\sigma\sqrt{T})-h)T}$ converges to $e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2}$, as $T \rightarrow \infty$. Furthermore, we have the estimate

$$\left| e^{(s(t/\sigma\sqrt{T})-h)T} - e^{-\langle t, \mathcal{D}t \rangle / 2\sigma^2} \right| \leq 2e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2}.$$

Hence, applying the Dominated Convergence Theorem, we obtain the desired result.

Lemma 5.

$$\lim_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} A_2(T, \alpha) = 0.$$

Proof. By Lemma 3(ii), for $t \notin U(\delta/\sigma\sqrt{T})$,

$$e^{-hT} \mathcal{S}_T(t/\sigma\sqrt{T}) = O(T^{-(k/2+1)}),$$

so that $\sup_{\alpha \in \mathbb{Z}^k} A_2(T, \alpha) = O(T^{-1})$.

Proof of Proposition 5. Combining the above results we have that

$$\lim_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \frac{h\sigma^k T^{k/2}}{e^{hT}} \psi(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1}\alpha \rangle / 2T} \right| \leq C \left\{ \int_{\mathbb{R}^k} e^{-\langle t, \mathcal{D}t \rangle / 4\sigma^2} dt \right\} \delta^2.$$

Since this holds for all sufficiently small $\delta > 0$, the proof is complete.

4. PROOF OF THEOREM 1

In this section we will use elementary arguments to deduce Theorem 1 from Proposition 5. Whenever we make a big- O estimate, the implied constant will be independent of α .

Write

$$\psi^*(T, \alpha) = \sum_{\substack{l(\gamma) \leq T \\ [\gamma] = \alpha}} l(\gamma).$$

An easy argument shows that

$$\psi(T, \alpha) = \psi^*(T, \alpha) + O(T^2 e^{hT/2}).$$

Thus, Proposition 5 implies the following.

Proposition 6.

$$\lim_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \frac{h\sigma^k T^{k/2}}{e^{hT}} \psi^*(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1} \alpha \rangle / 2T} \right| = 0.$$

Finally, we consider $\pi(T, \alpha)$. It is easy to see that

$$\psi^*(T, \alpha) \leq T\pi(T, \alpha).$$

For the corresponding lower bound, choose $\tau > 0$ and set $\theta = (1 + \tau)^{-1} < 1$. Then

$$\begin{aligned} \frac{T\pi(T, \alpha)}{e^{hT}} &= \frac{T}{e^{hT}} \sum_{\substack{\theta T < l(\gamma) \leq T \\ [\gamma] = \alpha}} 1 + \frac{T\pi(\theta T, \alpha)}{e^{hT}} \\ &\leq \frac{1 + \tau}{e^{hT}} \sum_{\substack{\theta T < l(\gamma) \leq T \\ [\gamma] = \alpha}} l(\gamma') + \frac{T\pi(\theta T, \alpha)}{e^{hT}} \\ &\leq \frac{(1 + \tau)\psi^*(T, \alpha)}{e^{hT}} + \frac{T\#\{\gamma : l(\gamma) \leq \theta T\}}{e^{hT}}. \end{aligned}$$

Using the estimate $\#\{\gamma : l(\gamma) \leq T\} = O(e^{hT}/T)$ [13], we have established

$$\begin{aligned} 0 &\leq \frac{T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{T^{k/2}}{e^{hT}} \psi^*(T, \alpha) \\ &\leq \frac{\tau T^{k/2}}{e^{hT}} \psi^*(T, \alpha) + O(T^{k/2} e^{(\theta-1)hT}), \end{aligned}$$

so that, by applying Proposition 6,

$$\limsup_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \frac{T^{k/2+1}}{e^{hT}} \pi(T, \alpha) - \frac{T^{k/2}}{e^{hT}} \psi^*(T, \alpha) \right| \leq \frac{\tau}{(2\pi)^{k/2} h \sigma^k}.$$

Since $\tau > 0$ is arbitrary, this proves Theorem 1.

5. HOMOLOGICALLY FULL ANOSOV FLOWS

The asymptotic identity (0.1) has been generalized to certain transitive Anosov flows $\phi_t : N \rightarrow N$, where N is a compact smooth Riemannian manifold. We now use γ to denote a (prime) periodic orbit of ϕ , with least period $l(\gamma)$. Once again, we write $[\gamma]$ for the torsion-free part of the homology class of γ in $H_1(N, \mathbb{Z}) \cong \mathbb{Z}^k \oplus \text{Tor}$. We say that ϕ is *homologically full* if every homology class in $H_1(N, \mathbb{Z})$ is represented

by a closed orbit. In this case, there exist $\xi \in H^1(N, \mathbb{R})$, $0 < h^* \leq h$ and $C_0 > 0$ such that

$$(5.1) \quad \pi(T, \alpha) \sim C_0 e^{-\langle \xi, \alpha \rangle} \frac{e^{h^* T}}{T^{k/2+1}}, \quad \text{as } T \rightarrow \infty.$$

This result was first proved in [19], drawing on ideas from [10]. An alternative proof was given in [2] and a more precise version is contained in [17]. In this section, we shall sketch a new proof of (5.1), using the techniques discussed above. However, we will not make any claims about uniformity.

Remark. We can define a function $\mathbf{p} : H^1(N, \mathbb{R}) \rightarrow \mathbb{R}$ by $\mathbf{p}([\omega]) = P(\omega(\mathcal{X}))$, where ω is a closed 1-form representing the cohomology class $[\omega]$ and \mathcal{X} is the vector field tangent to ϕ . Then ξ and h^* are characterized by the formulae

$$h^* = \mathbf{p}(\xi) = \min\{\mathbf{p}(\xi') : \xi' \in H^1(N, \mathbb{R})\}.$$

We begin by considering a modified family of L -functions. We define

$$(5.2) \quad L(s, t) = \prod_{\gamma} \left(1 - e^{-sl(\gamma) + \langle \xi, \alpha \rangle + i\langle t, [\gamma] \rangle}\right)^{-1},$$

which converges for $\operatorname{Re}(s) > h^*$. The extension of L'/L to a uniform strip, described in Propositions 1 and 2, is no longer valid; however, the next result provides a weaker substitute. As in the case of closed geodesics, an analysis due to Dolgopyat [5] is crucial here. For $\rho > 0$, write

$$\mathcal{R}(\rho) = \{s : \operatorname{Re}(s) > h^* - |\operatorname{Im}(s)|^{-\rho}, |\operatorname{Im}(s)| \geq 1\}.$$

Proposition 7 ([17]). *There exists a constant $\rho > 0$ such that, for all sufficiently small $\delta > 0$, the following statements are true.*

- (i) *There exists an analytic function $s : U(\delta) \rightarrow \{z \in \mathbb{C} : \operatorname{Re}(z) \leq h^*\}$, satisfying $s(0) = h^*$ and $\operatorname{Re}(s(t)) < h^*$ for $t \neq 0$ such that*

$$\frac{L'(s, t)}{L(s, t)} + \frac{1}{s - s(t)}$$

is analytic in $\mathcal{R}(\rho)$.

- (ii) *For $t \notin U(\delta)$, $L'(s, t)/L(s, t)$ is analytic in $\mathcal{R}(\rho)$.*

Proposition 8 ([17]). *There exist $C > 0$ and $\beta > 0$ such that, for all $t \in [-\pi, \pi]^k$,*

$$\left| \frac{L'(s, t)}{L(s, t)} \right| \leq C |\operatorname{Im}(s)|^\beta,$$

for $s \in \mathcal{R}(\rho)$.

Although the function $s(t)$ is now complex valued, it is still the case that $\nabla s(0) = 0$ and that $\nabla^2 s(0)$ is real and strictly negative definite. Moreover, again writing $\mathcal{D} = -\nabla^2 s(0)$ and $\sigma^{2k} = \det \mathcal{D}$, the function $e^{s(t) - h^*}$ still satisfies the conclusions of Proposition 3.

We shall now mimic the arguments of Section 2. However, the weaker bounds on $L'(s, t)/L(s, t)$ force us to use a more complicated auxiliary function. For $n \geq 0$, define

$$\psi_n(T, \alpha) = \frac{e^{\langle \xi, \alpha \rangle}}{n!} \sum'_{\substack{l(\gamma') \leq T \\ [\gamma'] = \alpha}} \Lambda(\gamma') \left(e^{h^* T} - e^{h^* l(\gamma')} \right)^n.$$

Then we have the identity

$$\sigma^k T^{k/2} \psi_n(T, \alpha) = \frac{1}{(2\pi)^k} \int_{I(\pi\sigma\sqrt{T})} e^{-i\langle t, \alpha \rangle / \sigma\sqrt{T}} \mathcal{S}_T^*(t/\sigma\sqrt{T}) dt,$$

where

$$\mathcal{S}_T^*(t) = \sum'_{l(\gamma') \leq T} \Lambda(\gamma') e^{\langle \xi, [\gamma'] \rangle + i\langle t, [\gamma'] \rangle} \left(e^{h^* T} - e^{h^* l(\gamma')} \right)^n.$$

In order to estimate the function $\mathcal{S}_T^*(t)$ we need the following identity, for $d > h^*$,

$$(5.3) \quad \mathcal{S}_T^*(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{(s+n)T}}{s(s+1)\cdots(s+n)} ds,$$

where we have used the formula

$$\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{y^s}{s(s+1)\cdots(s+n)} ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{n!} \left(1 - \frac{1}{y}\right)^n & \text{if } y \geq 1. \end{cases}$$

Choose $0 < \epsilon < 1/\rho$ and set $R = T^\epsilon$ and $d = h^* + T^{-1}$. Then replacing the integral in (5.3) with the truncated integral \int_{d-iR}^{d+iR} introduces an error of order $O(e^{(h^*+n)T}/T^{\epsilon n})$. Using the estimates

(a)

$$\left| \left(\int_{h^*-R^{-\rho}+iR}^{d+iR} + \int_{h^*-R^{-\rho}-iR}^{d-iR} \right) \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{(s+n)T}}{s(s+1)\cdots(s+n)} ds \right| \\ = O(R^{\beta-\rho-n-1} e^{(h^*+n)T}) = O(e^{(h^*+n)T} T^{-\epsilon(\rho+n+1-\beta)});$$

(b)

$$\left| \int_{h^*-R^{-\rho}\pm i}^{h^*-R^{-\rho}\pm iR} \left(-\frac{L'(s, t)}{L(s, t)} \right) \frac{e^{(s+n)T}}{s(s+1)\cdots(s+n)} ds \right| \\ = O(R^\beta e^{(h^*+n-R^{-\rho})T}) = O(T^{\beta\epsilon} e^{(h^*+n)T} e^{-T^{1-\epsilon\rho}}),$$

we may repeat the proof of Lemma 3 to obtain the following lemma.

Lemma 6. *Setting $N = \min\{\epsilon n, \epsilon(\rho + n + 1 - \beta)\}$, we have the following. (The implied constants are independent of t .)*

(i) For $t \in U(\delta)$,

$$\mathcal{S}_T^*(t) = \frac{e^{(s(t)+n)T}}{s(t)(s(t)+1)\cdots(s(t)+n)} + O\left(\frac{e^{(h^*+n)T}}{T^N}\right).$$

(ii) For $t \notin U(\delta)$, $\mathcal{S}_T^*(t) = O(e^{(h^*+n)T}/T^N)$.

Provided n is sufficiently large such that $N > k/2$, we may repeat the arguments used in the proof of Proposition 5 to obtain

$$\lim_{T \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}^k} \left| \prod_{j=0}^n (h^* + j) \frac{\sigma^k T^{k/2}}{e^{(h^*+n)T}} \psi_n(T, \alpha) - \frac{1}{(2\pi)^{k/2}} e^{-\langle \alpha, \mathcal{D}^{-1} \alpha \rangle / 2T} \right| = 0.$$

From this it immediately follows that

$$\psi_n(T, \alpha) \sim \frac{1}{(2\pi)^{k/2} \sigma^k} \prod_{j=0}^n \frac{1}{(h^* + j)} \frac{e^{(h^*+n)T}}{T^{k/2}}, \quad \text{as } T \rightarrow \infty.$$

The asymptotic formula

$$\psi_0(T, \alpha) \sim \frac{1}{(2\pi)^{k/2} h^* \sigma^k} \frac{e^{h^* T}}{T^{k/2}}, \quad \text{as } T \rightarrow \infty,$$

now follows by a standard inductive argument (cf. p. 35 of [7]). Finally, (5.1) may be deduced as in Section 3. (Note that one needs the *a priori* estimate $\limsup_{T \rightarrow \infty} (\pi(T, \alpha))^{1/T} \leq e^{h^*}$, which follows from the convergence of (5.2) for $\operatorname{Re}(s) > h^*$.)

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